Direct Methods for Stability Analysis of Electric Power Systems

Theoretical Foundation, BCU Methodologies, and Applications

Hsiao-Dong Chiang

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Preface

Power system instabilities are unacceptable to society. Indeed, recent major blackouts in North America and in Europe have vividly demonstrated that power interruptions, grid congestions, or blackouts significantly impact the economy and society. At present, stability analysis programs routinely used in utilities around the world are based mostly on step-by-step numerical integrations of power system stability models to simulate system dynamic behaviors. This off-line practice is inadequate to deal with current operating environments and calls for online evaluations of changing overall system conditions.

Several significant benefits and potential applications are expected from the movement of transient stability analysis from the off-line mode to the online operating environment. However, this movement is a challenging task and requires several breakthroughs in measurement systems, analytical tools, computation methods, and control schemes. An alternate approach to transient stability analysis employing energy functions is called the direct method, or termed the energy function-based direct method. Direct methods offer several distinctive advantages. For example, they can determine transient stability without the time-consuming numerical integration of a (postfault) power system. In addition to their speed, direct methods can provide useful information regarding the derivation of preventive control and enhancement control actions for power system stability.

Direct methods have a long developmental history spanning six decades. Despite the fact that significant progress has been made, direct methods have been considered impractical by many researchers and users. Several challenges and limitations must be overcome before direct methods can become a practical tool. This book seeks to address these challenges and limitations.

The main purpose of this book is to present a comprehensive theoretical foundation for the direct methods and to develop comprehensive BCU solution methodologies along with their theoretical foundations. In addition, a comprehensive energy function theory, which is an extension of the Lyapunov function theory, is presented along with general procedures for constructing numerical energy functions for general power system transient stability models. It is believed that solving challenging practical problems efficiently can be accomplished through a thorough understanding of the underlying theory, in conjunction with exploring the special features of the practical problem under study to develop effective solution methodologies.

There are 25 chapters contained in this book. These chapters are classified into the following subjects:
The following stages of research and development can lead to fruitful and practical applications:

**Stage 1.** Development of theoretical foundations

**Stage 2.** Development of the solution methodology

**Stage 3.** Development of reliable methods to numerically implement the solution methodology

**Stage 4.** Software implementation and evaluation

**Stage 5.** Industry user interactions

**Stage 6.** Practical system installation

The first three stages are suitable for university and research institution application, while the last four stages are more suitable for commercial entities. This text focuses on Stages 1 and 2 and touches upon Stage 3. In the following volume, Stage 3 will be more thoroughly explored along with Stages 4 through 6.

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Hsiao-Dong Chang
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H-D. C.
Chapter 1

Introduction and Overview

1.1 INTRODUCTION

Power system instabilities are unacceptable to society. Indeed, recent major blackouts in North America and in Europe have vividly demonstrated that power interruptions, grid congestions, or blackouts significantly impact the economy and society. In August 1996, disturbances cascaded through the West Coast transmission system, causing widespread blackouts that cost an estimated $2 billion and left 12 million customers without electricity for up to 8 h. In June 1998, transmission system constraints disrupted the wholesale power market in the Midwest, causing price rises from an average of $30 per megawatt hour to peaks as high as $10,000 per megawatt hour. Similar price spikes also occurred in the summers of 1999 and 2000. In 2003, the Northeast blackout left 50 million customers without electricity and the financial loss was estimated at $6 billion. According to a research firm, the annual cost of power outages and fluctuations worldwide was estimated to be between $119 and $188 billion yearly. Power outages and interruptions clearly have significant economic consequences for society.

The ever-increasing loading of transmission networks coupled with a steady increase in load demands has pushed the operating conditions of many worldwide power systems ever closer to their stability limits. The combination of limited investment in new transmission and generation facilities, new regulatory requirements for transmission open access, and environmental concerns are forcing transmission networks to carry more power than they were designed to withstand. This problem of reduced operating security margins is further compounded by factors such as (1) the increasing number of bulk power interchange transactions and non-utility generators, (2) the trend towards installing higher-output generators with lower inertia constants and higher short circuit ratios, and (3) the increasing amount of renewable energies. Under these conditions, it is now well recognized that any violation of power system dynamic security limits leads to far-reaching consequences for the entire power system.
Chapter 1  Introduction and Overview

By nature, a power system continually experiences two types of disturbances: *event disturbances* and *load variations*. Event disturbances (contingencies) include loss of generating units or transmission components (lines, transformers, and substations) due to short circuits caused by lightning, high winds, and failures such as incorrect relay operations, insulation breakdowns, sudden large load changes, or a combination of such events. Event disturbances usually lead to a change in the network configuration of the power system due to actions from protective relays and circuit breakers. They can occur as a single equipment (or component) outage or as multiple simultaneous outages when taking relay actions into account. Load variations are variations in load demands at buses and/or power transfers among buses. The network configuration may remain unchanged after load variations. Power systems are planned and operated to withstand certain disturbances. The North American Electric Reliability Council defines security as the ability to prevent cascading outages when the bulk power supply is subjected to severe disturbances. Individual reliability councils establish the types of disturbances that their systems must withstand without cascading outages.

A major activity in power system planning and operation is the examination of the impact a set of credible disturbances has on a power system’s dynamic behavior such as stability. Power system stability analysis is concerned with a power system’s ability to reach an acceptable steady state (operating condition) following a disturbance. For operational purposes, power system stability analysis plays an important role in determining the system operating limits and operating guidelines. During the planning stage, power system stability analysis is performed to assess the need for additional facilities and the locations at which additional control devices to enhance the system’s static and dynamic security should be placed. Stability analysis is also performed to check relay settings and to set the parameters of control devices. Important conclusions and decisions about power system operations and planning are made based on the results of stability studies.

Transient stability problems, a class of power system stability problems, have been a major operating constraint in regions that rely on long-distance transfers of bulk power (e.g., in most parts of the Western Interconnection in the United States, Hydro-Québec, the interfaces between the Ontario/New York area and the Manitoba/Minnesota area, and in certain parts of China and Brazil). The trend now is that many parts of the various interconnected systems are becoming constrained by transient stability limitations. The wave of recent changes has caused an increase in the adverse effects of both event disturbances and load variations in power system stability. Hence, it is imperative to develop powerful tools to examine power system stability in a timely and accurate manner and to derive necessary control actions for both preventive and enhancement control.

1.2 TRENDS OF OPERATING ENVIRONMENT

The aging power grid is vulnerable to power system disturbances. Many transformers in the grid approach or surpass their design life. The transmission system
is often under-invested and overstrained. These result in vulnerable power grids constantly operating near their operating limits. In addition, this operating environment encounters more challenges brought about by dispersed generations whose prime movers can be any renewable energy source such as wind power. As is well recognized, these small-size dispersed generation systems raise even greater concerns of power system stability. Hence, with current power system operating environments, it is increasingly difficult for power system operators to generate all the operating limits for all possible operating conditions under a list of credible contingencies.

At present, most energy management systems periodically perform online power system static security assessment (SSA) and control to ensure that the power system can withstand a set of credible contingencies. The assessment involves selecting a set of credible contingencies and evaluating the system’s response to those contingencies. Various software packages for security assessment and control have been implemented in modern energy control centers. These packages provide comprehensive online security analysis and control based almost exclusively on steady-state analysis, making them applicable to SSA and control but not to online transient stability assessment (TSA). Instead, off-line transient stability analysis has been performed for postulated operating conditions. The turn-around time for a typical study can range from hours to days depending on the number of postulated operating conditions and the dynamic study period of each contingency. This off-line practice is inadequate to deal with current operating environments and calls for online evaluations of the constantly changing overall system conditions.

The lack of performing online TSAs in an energy management system can have serious consequences. Indeed, any violation of dynamic security limits has far-reaching impacts on the entire power system and thus on the society. From a financial viewpoint, the costs associated with a power outage can be tremendous. Online dynamic security assessment is an important tool for avoiding dynamic security limit violations. It is fair to say that the more stressed a power system, the stronger the need for online dynamic security assessments.

Several significant benefits and potential applications are expected from the movement of transient stability analysis from the off-line mode to the online operating environment. The first benefit is that a power system can be operated with operating margins reduced by a factor of 10 or more if the dynamic security assessment is based on the actual system configuration and actual operating conditions instead of assumed worst-case conditions, as is done in off-line studies. This ability is especially significant since current environments have pushed power systems to operate with low reserve margins closer to their stability limits. A second benefit to online analysis is that the large number of credible contingencies that needs to be assessed can be reduced to those contingencies relevant to actual operating conditions. Important consequences obtained from this benefit are that more accurate operating margins can be determined and more power transfers among different areas, or different zones of power networks, can be realized. Compared to off-line studies, online studies require much less engineering resources, thereby freeing these resources for other critical activities.
1.3 ONLINE TSA

Online TSA is designed to provide system operators with critical system stability information including (1) TSA of the current operating condition subject to a list of contingencies and (2) available (power) transfer limits at key interfaces subject to transient stability constraints. A complete online TSA assessment cycle is typically in the order of minutes, say, 5 min. This cycle starts when all necessary data are available to the system and ends when the system is ready for the next cycle. Depending on the size of the underlying power systems, it is estimated that, for a large-size power system such as a 15,000-bus power system, the number of contingencies in a contingency list is between 2000 and 3000. The contingency types will include both a three-phase fault with primary clearance and a single line-to-ground fault with backup clearance.

When a cycle of online TSA is initiated, a list of credible contingencies, along with information from the state estimator and topological analysis, is applied to the online TSA program whose basic function is to identify unstable contingencies from the contingency list. An operating condition is said to be transiently stable if the contingency list contains no unstable contingencies; otherwise, it is transiently unstable. The task of online TSA, however, is very challenging.

The strategy of using an effective scheme to screen out a large number of stable contingencies, capture critical contingencies, and apply detailed simulation programs only to potentially unstable contingencies is well recognized. This strategy has been successfully implemented in online SSA. The ability to screen several hundred contingencies to capture tens of the critical contingencies has made the online SSA feasible. This strategy can be applied to online TSA. Given a set of credible contingencies, the strategy would break the task of online TSA into two stages of assessments (Chadalavada et al., 1997; Chiang et al., 1997):

**Step 1.** Perform the task of dynamic contingency screening to quickly screen out contingencies that are definitely stable from a set of credible contingencies.

**Step 2.** Perform detailed assessment of dynamic performance for each contingency remaining in Stage 1.

Dynamic contingency screening is a fundamental function of an online TSA system. The overall computational speed of an online TSA system depends greatly on the effectiveness of the dynamic contingency screening, the objective of which is to identify contingencies that are definitely stable and thereby to avoid further stability analysis for these contingencies. It is due to the definite classification of stable contingencies that considerable speedup can be achieved for TSA. Contingencies that are either undecided or identified as critical or unstable are then sent to the time-domain transient stability simulation program for further stability analysis.

Online TSA can provide an accurate determination of online transfer capability constrained by transient stability limits. This accurate calculation of transfer capability allows remote generators with low production cost to be economically dispatched...
1.4 Need for New Tools

to serve load centers. We consider a hypothetical power system containing a remote
generator with low production cost, say, a hydro generator of $2 per megawatt hour
and a local generator with a high production cost of $5 per megawatt hour that all
supply electricity to a load center of 2500 MW (see Figure 1.1). According to the
off-line analysis, the transfer capability between the remote generator and the load
center was 2105 MW. With a 5% security margin, the output of the remote generator
was set to 2000 MW. The local generator then needs to supply 500 MW to the load
center to meet the load demand. On the other hand, the actual transfer capability
between the remote generator and the load center, according to online TSA, was
2526 MW instead of 2105 MW. With a 5% security margin, the output of the remote
generator was set to 2400 MW, while the output of the local generator was set to
100 MW to meet the load demand. By comparing these two different schemes of
real power dispatch based on two different transfer capability calculations, the dif-
ference in production cost is about $1200 per hour or $28,800 per day. It can be
observed that even for such a relatively small load demand of 2500 MW, online TSA
allows for significant financial savings amounting to about $10.5 million per year.
We recognize that practical power systems may not resemble this hypothetical power
system; however, it does illustrate the significant financial benefits of online TSA.

1.4 NEED FOR NEW TOOLS

At present, stability analysis programs routinely used in utilities around the world
are based mostly on step-by-step numerical integrations of power system stability
models used to simulate system dynamic behaviors. This practice of power system
stability analysis based on the time–domain approach has a long history. The

Figure 1.1 A hypothetical power system and analysis of financial savings.
stability of the postfault system is assessed based on simulated postfault trajectories. The typical simulation period for the postfault system is 10 s and can go beyond 15 s if multiswing instability is of concern, making this conventional approach rather time-consuming.

The traditional time-domain simulation approach has several disadvantages. First, it requires intensive, time-consuming computation efforts; therefore, it has not been suitable for online application. Second, it does not provide information as to how to derive preventive control when the system is deemed unstable nor how to derive enhancement control when the system is deemed critically stable, and finally, it does not provide information regarding the degree of stability (when the system is stable) and the degree of instability (when the system is unstable) of a power system. This information is valuable for both power system planning and operation.

From a computational viewpoint, online TSA involves solving a large set of mathematical models, which is described by a large set of nonlinear differential equations in addition to the nonlinear algebraic equations involved in the SSA. For a 14,000-bus power system transient stability model, one dynamic contingency analysis can involve solving a set of 15,000 differential equations and 40,000 nonlinear algebraic equations for a time duration of 10–20 s in order to assess the power system stability under the study contingency. Online TSA requires the ability to analyze hundreds or even thousands of contingencies every 5–10 min using online data and system state estimation results. Thus, the traditional time-domain simulation approach cannot meet this requirement.

The computational effort required by online TSA is roughly three magnitudes higher than that of the SSA. This explains why TSA has long remained an off-line activity instead of an online activity in the energy management system. Extending the functions of energy management systems to take into account online TSA and control is a challenging task and requires several breakthroughs in measurement systems, analytical tools, computation methods, and control schemes.

1.5 DIRECT METHODS: LIMITATIONS AND CHALLENGES

An alternate approach to transient stability analysis employing energy functions, called direct methods, or termed energy function-based direct methods, was originally proposed by Magnusson (1947) in the late 1940s and was pursued in the 1950s by Aylett (1958). Direct methods have a long developmental history spanning six decades. Significant progress, however, has been made only recently in the practical application of direct methods to transient stability analysis. Direct methods can determine transient stability without the time-consuming numerical integration of a (postfault) power system. In addition to their speed, direct methods also provide a quantitative measure of the degree of system stability. This additional information makes direct methods very attractive when the relative stability of different network configuration plans must be compared or when system operating limits constrained by transient stability must be calculated quickly. Another advantage to direct methods
is that they provide useful information regarding the derivation of preventive control actions when the underlying power system is deemed unstable and the derivation of enhancement control actions when the underlying power system is deemed critically stable.

Despite the fact that significant progress has been made in energy function-based direct methods over the last several decades, they have been considered impractical by many researchers and users for power system applications. Indeed, direct methods must overcome several challenges and limitations before they can become a practical tool.

From an analytical viewpoint, direct methods were originally developed for power systems with autonomous postfault systems. As such, there are several challenges and limitations involved in the practical applications of direct methods for power system transient stability analysis, some of which are inherent to these methods while others are related to their applicability to power system models. These challenges and limitations can be classified as follows:

**Challenges**
- The modeling challenge
- The function challenge
- The reliability challenge

**Limitations**
- The scenario limitation
- The condition limitation
- The accuracy limitation

The modeling challenge stems from the requirement that there exists an energy function for the (postfault) transient stability model of study. However, the problem is that not every (postfault) transient stability model admits an energy function; consequently, simplified transient stability models have been used in direct methods. A major shortcoming of direct methods in the past has been the simplicity of the models they can handle. Recent work in this area has made significant advances. The current progress in this direction is that a general procedure of constructing numerical energy functions for complex transient stability models is available. This book will devote Chapters 6 and 7 to this topic.

The function limitation stipulates that direct methods are only applicable to first swing stability analysis of power system transient stability models described by pure differential equations. Recent work in the development of the controlling UEP method has extended the first-swing stability analysis into a multiswing stability analysis. In addition, the controlling UEP method is applicable to power system transient stability models described by differential and algebraic equations. This book will devote Chapters 11 through 13 to this topic.

The scenario limitation for direct methods comes from the requirement that the initial condition of a study postfault system must be available and the requirement
that the postfault system must be autonomous. It is owing to the requirement of the availability of the initial condition that makes numerical integration of the study fault-on system a must for direct methods. Hence, the initial condition of a study postfault system can only be obtained via the time-domain approach and cannot be available beforehand. On the other hand, the requirement that the postfault system be autonomous imposes the condition that the fault sequence on the system must be well-defined in advance. Currently, the limitation that the postfault system must be an autonomous dynamical system is partially removed. In particular, the postfault system does not need to be a “pure” autonomous system and it can be constituted by a series of autonomous dynamical systems.

The condition limitation is an analytical concern related to the required conditions for postfault power systems: a postfault stable equilibrium point must exist and the prefault stable equilibrium point must lie inside the stability region of the postfault stable equilibrium point. This limitation is inherent to the foundation of direct methods. Generally speaking, these required conditions are satisfied on stable contingencies, while they may not be satisfied on unstable contingencies. From an application viewpoint, this condition limitation is a minor concern and direct methods can be developed to overcome this limitation.

The accuracy limitation stems from the fact that analytical energy functions for general power system transient stability models do not exist. Regarding the accuracy limitation, it has been observed in numerous studies that the controlling UEP method, in conjunction with appropriate numerical energy functions, yields accurate stability assessments. Numerical energy functions are practically useful in direct methods. In this book, methods and procedures to construct accurate numerical energy functions will be presented.

The reliability challenge is related to the reliability of a computational method in computing the controlling UEP for every study contingency. From a theoretical viewpoint, this text will demonstrate the existence and uniqueness of the controlling UEP with respect to a fault-on trajectory. Furthermore, the controlling UEP is independent of the energy function used in the direct stability assessment. Hence, the task of constructing an energy function and the task of computing the controlling UEP are not interrelational. From a computational viewpoint, the task of computing the controlling UEP is very challenging. We will present in Chapter 12 the computational challenges in computing the controlling UEP. A total of seven challenges in computing the controlling UEP will be highlighted. These challenges call into doubt the correctness of any attempt to directly compute the controlling UEP of the original power system stability model. This analysis serves to explain why previous methods proposed in the literature fail to compute the controlling UEP.

The above analysis reveals three important implications for the development of a reliable numerical method for computing controlling UEPs:

1. These computational challenges should be taken into account in the development of numerical methods for computing the controlling UEP.

2. It is impossible to directly compute the controlling UEP of a power system stability model without using the iterative time-domain method.
3. It is possible to directly compute the controlling UEP of an artificial, reduced-state power system stability model without using the iterative time-domain method.

In this book, it will be shown that it is fruitful to develop a tailored solution algorithm for finding the controlling UEPs by exploiting special properties as well as some physical and mathematical insights into the underlying power system stability model. We will discuss in great detail such a systematic method, called the BCU method, for finding controlling UEPs for power system models in Chapters 14 through 17. The BCU method does not attempt to directly compute the controlling UEP of a power system stability model (original model); instead, it computes the controlling UEP of a reduced-state model and relates the computed controlling UEP to the controlling UEP of the original model. This book will devote Chapters 14 through 24 to present the following family of BCU methods:

- The BCU method
- The BCU–exit point method
- The group-based BCU–exit point method
- The group-based BCU–CUEP method
- The group-based BCU method

This book will also explain how to develop tailored solution methodologies by exploring special properties as well as some physical and mathematical insights into the underlying power system stability model. For instance, it will be explained how the group properties of contingencies in power systems are discovered. These group properties will be explored and incorporated into the development of a group-based BCU method. This exploration of group properties leads to a significant reduction in computational efforts for reliably computing controlling UEPs for a group of coherent contingencies and to the development of effective preventive control actions against a set of insecure contingencies and enhancement control actions for a set of critical contingencies.

1.6 PURPOSES OF THIS BOOK

The main purpose of this book is to present a comprehensive theoretical foundation for direct methods and to develop comprehensive BCU solution methodologies along with their theoretical foundations. BCU methodologies have been developed to reliably compute controlling UEPs and to reliably compute accurate critical values, which are essential pieces of information needed in the controlling UEP method. In addition, a comprehensive energy function theory, which is an extension of the Lyapunov function theory, is presented along with a general procedure for constructing numerical energy functions for general power system transient stability models.

This author believes that solving challenging practical problems efficiently can be accomplished through a thorough understanding of the underlying theory, in
conjunction with exploring the special features of the practical problem under study, to develop effective solution methodologies. This book covers both a comprehensive theoretical foundation for direct methods and comprehensive BCU solution methodologies. There are 25 chapters contained in this book. These chapters can be classified into the following (see Figure 1.2):

Chapter 2: System Modeling and Stability Problems

Figure 1.2 An overview of the organization and content of this book.
1.6 Purposes of This Book

Chapter 14
Chapter 15
Chapter 16
Chapter 17
Chapter 18
Chapter 19
Chapter 20
Chapter 21
Chapter 22
Chapter 23
Chapter 24
Chapter 25

BCU Methods: Theoretical Foundation
Numerical BCU Methods
Analytical and Numerical Justification of the BCU Method
BCU–Exit Point Method
Group Properties of Power Systems
Group-Based BCU Methods
Perspectives and Future Directions

Figure 1.2 Continued

Theory of Stability Regions
Chapter 3: Lyapunov Stability and Stability Regions of Nonlinear Dynamical Systems
Chapter 4: Quasi-Stability Regions: Analysis and Characterization
In summary, this book presents the following theoretical developments as well as solution methodologies with a focus on practical applications for the direct analysis of large-scale power system transient stability; in particular, this book

- provides a general framework for general direct methods, particularly the controlling UEP method;
- develops a comprehensive theoretical foundation for the controlling UEP method, the potential energy boundary surface (PEBS) method, and the closest UEP method;
1.6 Purposes of This Book

- presents the BCU methodologies, including the network-reduction BCU method and the network-preserving BCU method;
- presents the theoretical foundation for both the network-reduction BCU method and the network-preserving BCU method;
- develops numerical implementations of both the network-reduction BCU method and the network-preserving BCU method;
- demonstrates the computational procedure of numerical BCU methods using the stability boundary of the original system model and that of the reduced-state model;
- conducts analytical studies of the transversality condition of the BCU method and relates the transversality condition with the boundary condition;
- presents the BCU–exit point method;
- develops group properties of power system contingencies;
- explores the static and dynamic group properties of power system coherent contingencies;
- develops the group-based BCU–exit point method and the group-based BCU–CUEP method; and
- develops group-based BCU methodologies, including the group-based BCU–exit point method, the group-based BCU–CUEP method, and the group-based BCU method.
Chapter 2

System Modeling and Stability Problems

Electric power systems are nonlinear in nature. Their nonlinear behaviors are difficult to predict due to (1) the extraordinary size of the systems, (2) the nonlinearity in the systems, (3) the dynamic interactions within the systems, and (4) the complexity of component modeling. These complicating factors have forced power system engineers to analyze the complicated behaviors of power systems through the process of modeling, simulation, analysis, and validation.

2.1 INTRODUCTION

The complete power system model for calculating system dynamic response relative to a disturbance comprises a set of first-order differential equations:

\[ \dot{x} = f(x, y, u), \quad (2.1) \]

describing the internal dynamics of devices such as generators, their associated control systems, certain loads, and other dynamically modeled components. The model is also comprised of a set of algebraic equations,

\[ 0 = g(x, y, u), \quad (2.2) \]

describing the electrical transmission system (the interconnections between the dynamic devices) and the internal static behaviors of passive devices (such as static loads, shunt capacitors, fixed transformers, and phase shifters). The differential equation (Eq. 2.1) typically describes the dynamics of the speed and angle of generator rotors; the flux behaviors in generators; the response of generator control systems such as excitation systems, voltage regulators, turbines, governors, and boilers; the dynamics of equipment such as synchronous VAR compensators (SVCs), DC lines, and their control systems; and the dynamics of dynamically modeled loads such as induction motors. The stated variables \( x \) typically include generator rotor angles, generator velocity deviations (speeds), mechanical powers, field voltages, power...
system stabilizer signals, various control system internal variables, and voltages and angles at load buses (if dynamic load models are employed at these buses). The algebraic equations (Eq. 2.2) are composed of the stator equations for each generator, the network equations of transmission networks and loads, and the equations defining the feedback stator quantities. An aggregated representation of each local distribution network is usually used in simulating power system dynamic behaviors. The forcing functions $u$ acting on the differential equations are terminal voltage magnitudes, generator electrical powers, signals from boilers, automatic generation control systems, and so on.

Some control system internal variables have upper bounds on their values due to their physical saturation effects. Let $z$ be the vector of these constrained state variables; then, the saturation effects can be expressed as

$$0 < z(t) \leq \bar{z}.$$  

(2.3)

For a 900-generator, 14,000-bus power system, the number of differential equations can easily reach as many as 20,000, while the number of nonlinear algebraic equations can easily reach as many as 32,000. The sets of differential equations (Eq. 2.1) are usually loosely coupled (Kundur, 1994; Stott, 1979; Tanaka et al., 1994).

### 2.2 POWER SYSTEM STABILITY PROBLEM

By nature, a power system continually experiences two types of disturbances: event disturbances and load disturbances (Anderson and Fouad, 2003; Balu et al., 1992). Event disturbances include loss of generating units or transmission components (lines, transformers, and substations) due to short circuits caused by lightning, high winds, failures such as incorrect relay operations or insulation breakdown, or a combination of such events. Event disturbances usually lead to a change in the configuration of power networks. Load disturbances, on the other hand, are the sudden large load changes and the small random fluctuations in load demands. The configuration of power networks usually remains unchanged after load disturbances.

To protect power systems from damage due to disturbances, protective relays are placed strategically throughout a power system to detect faults (disturbances) and to trigger the opening of circuit breakers necessary to isolate faults. These relays are designed to detect defective lines and apparatus or other power system conditions of an abnormal or dangerous nature and to initiate appropriate control actions. Due to the action of these protective relays, a power system subject to an event disturbance can be viewed as going through network configuration changes in three stages: the prefault, the fault-on, and the postfault systems (see Table 2.1).

The prefault system is in a stable steady state; when an event disturbance occurs, the system then moves into the fault-on system before it is cleared by protective system operations. Stated more formally, in the prefault regime, the system is at a known stable equilibrium point (SEP), say $(x_{p}^{\infty}, y_{p}^{\infty})$. At some time $t_0$, the system undergoes a fault (an event disturbance), which results in a structural change in the system due to actions from relay and circuit breakers. Suppose the fault duration is
Chapter 2 System Modeling and Stability Problems

Table 2.1 The Time Evolution, System Evolution, Physical Mechanism, and Mathematical Descriptions of the Power System Stability Problem during the Prefault, Fault-On, and Postfault Stages

<table>
<thead>
<tr>
<th>Physical mechanism</th>
<th>System evolution and time evolution</th>
<th>Mathematical description</th>
</tr>
</thead>
<tbody>
<tr>
<td>System is operated around a stable equilibrium point.</td>
<td>Prefault system $t &lt; t_0$</td>
<td>$x(t), y(t)$, $(x_i, y_i)$</td>
</tr>
<tr>
<td>A fault occurs on the system that initiates relay actions and circuit breaker actions.</td>
<td>Fault-on system $t_0 \leq t \leq t_{cl}$</td>
<td>$\dot{x} = f^c_i(x, y)$, $0 = g^c_i(x, y)$, $t_0 \leq t \leq t_{F,i}$</td>
</tr>
<tr>
<td>The fault is cleared as the actions of circuit breakers are finished.</td>
<td>Postfault system $t &gt; t_{cl}$</td>
<td>$\dot{x} = f(x, y)$, $0 = g(x, y)$</td>
</tr>
</tbody>
</table>

confined to the time interval $[t_0, t_{cl}]$. During this interval, the fault-on system is described by (for ease of exposition, the saturation effects expressed as $0 < z(t) \leq \bar{z}$ are neglected in the following) the following set of differential and algebraic equations (DAEs):

$$\dot{x} = f^c(x, y), \quad t_0 \leq t \leq t_{cl}$$

$$0 = g^c(x, y),$$

where $x(t)$ is the vector of state variables of the system at time $t$. Sometimes, the fault-on system may involve more than one action from system relays and circuit breakers. In these cases, the fault-on systems are described by several sets of DAEs:

$$\dot{x} = f^c_1(x, y), \quad t_0 \leq t \leq t_{F,1}$$

$$0 = g^c_1(x, y)$$

$$\dot{x} = f^c_2(x, y), \quad t_{F,1} \leq t \leq t_{F,2}$$

$$0 = g^c_2(x, y)$$

$$\dot{x} = f^c_3(x, y), \quad t_{F,2} \leq t \leq t_{cl}$$

$$0 = g^c_3(x, y).$$

The number of sets of DAEs equals the number of separate actions due to system relays and circuit breakers. Each set of DAE depicts the system dynamics due to
one action from relays and circuit breakers. Suppose the fault is cleared at time $t_{cl}$ and no additional protective actions occur after $t_{cl}$. The system, termed the postfault system, is henceforth governed by postfault dynamics described by

$$\dot{x} = f_{PF}(x, y), \quad t_{cl} \leq t < \infty$$

$$0 = g_{PF}(x, y).$$

(2.6)

The network configuration may or may not be the same as the prefault configuration in the postfault system. We will use the notation $z(t_{cl}) = (x(t_{cl}), y(t_{cl}))$ to denote the fault-on state at switching time $t_{cl}$. The postfault trajectory after an event disturbance is the solution of Equation 2.6, with initial condition $z(t_{cl}) = (x(t_{cl}), y(t_{cl}))$ over the postfault time period $t_{cl} \leq t < t_{w}$.

The fundamental problem of power system stability due to a fault (i.e., a contingency) can be summarized as follows: given a prefault SEP and a fault-on system, will the postfault trajectory settle down to an acceptable steady state? A straightforward approach to assess the power system stability is to numerically simulate the system trajectory and then to examine whether the postfault trajectory settles down to an acceptable steady state. A simulated system trajectory of a large-scale power system transient stability model is shown in Figures 2.1 and 2.2. The simulated system trajectory is composed of the predisturbance trajectory (a SEP) and the fault-on trajectory and the postdisturbance (i.e., the postfault) trajectory. The simulated postfault trajectory settles down to a postfault SEP.

Power system dynamic behaviors after a contingency can be fairly complex. This is because electric power systems are composed of a large number of components (equipment and control devices) interacting with each other, exhibiting nonlinear dynamic behaviors with a wide range of timescales. For instance, the difference between the time constants of excitation systems and those of governors is roughly a couple of orders of magnitude. These physical differences are reflected in the
underlying differential equations, which contain variables of considerably different timescales. The dynamic behavior after a disturbance involves all of the system components, in varying degrees. For instance, a short circuit occurring on a transmission line will trigger the opening of circuit breakers to isolate the fault. This will cause variations in generator rotor speeds, bus voltages, and power flows through transmission lines. Depending on their individual characteristics, voltage variations will activate generator excitation system underload tap changer (ULTC) transformers, SVCs, and undervoltage relays and will cause changes in voltage-dependent loads. Meanwhile, speed variations will activate prime mover governors, underfrequency relays, and frequency-dependent loads. The variations of power flows will activate generation control and ULTC phase shifters. The degree of involvement from each component can be explored to determine the appropriate system model necessary for simulating the dynamic behaviors.

Traditional practice in power system analysis has been to use the simplest acceptable system model, which captures the essence of the phenomenon under study. For instance, the effect of a system component or a control device can be neglected when the timescale of its response is very small or very large compared to the time period of interest. The effects of these components can be roughly taken into account as follows: the dynamic behavior of a system component or a control device can be considered as instantaneously fast if the timescale of its response is very small as compared to the time period of interest. Likewise, the dynamic behavior of a system component or a control device can be considered as a constant if the timescale of its response is very large as compared to the time period of interest. This philosophy has been deemed acceptable because of the severe complexity.

Figure 2.2 The simulated dynamic behavior, prefault, fault-on, and postfault of a voltage magnitude of a large power system model. During the fault, the voltage magnitude drops to about 0.888 p.u.
involved with a full, large-scale power system model (Kundur, 1994; Stott, 1979; Tanaka et al., 1994).

Power system stability models have been divided into three types of stability models with different timescales: (1) a short-term stability model (predominately describing electromechanical transients) on which transient stability analysis is based, (2) a midterm stability model, and (3) a long-term stability model on which long-term stability analysis is based. This division of power system stability models is based on the different timescale involvement of each component and the control device on the overall system's dynamic behaviors (Cate et al., 1984; Kundur, 1994). These three models are described by a set of differential–algebraic equations of the same nature as Equations 2.1 and 2.2, but with different sets of state variables with different time constants. There is, however, a fuzzy boundary distinguishing between the midterm and long-term models. Compared with transient stability analysis, midterm and long-term dynamic behaviors have only come under study relatively recently (Chow, 1982; Kundur, 1994; Stott, 1979; Stubbe et al., 1989; Tanaka et al., 1994).

The time frame of electromechanical oscillations in rotor angle stability typically ranges from a few seconds to tens of seconds. The dynamics of excitation systems, automatic voltage regulators, SVCs, underfrequency load shedding, and undervoltage load shedding are all active in this time frame. These dynamics are called transient (short-term) dynamics, which extend over time intervals on the order of 10s. The adjective “transient” is added to angle stability to form the term “transient angle stability” when the transient (short-term) power system model is used in a simulation. Similarly, the “adjective transient” is added to voltage stability to form the term “transient voltage stability” when the short-term model is used in voltage stability analysis. When the transient dynamics subside, the system enters the midterm time frame, typically within several minutes, in which the dynamics from such components as ULTC, generator limiters, and load dynamics become active. The time frame following the midterm time frame is the long-term time frame in which turbines, prime mover governors, are active. The adjective “long-term” is added to angle (or voltage) stability to form the term long-term angle (or voltage) stability when the long-term model is used in the simulation.

For transient stability analysis, the assumption of one unique frequency is kept for the transmission network model, but generators have different speeds. Generators are modeled in greater detail, with shorter time constants compared with the models used in long-term stability analysis. Roughly speaking, transient stability models reflect the fast-varying system electrical components and machine angles and frequencies, while the long-term models are concerned with the representation of the slow oscillatory power balance, assuming that the rapid electrical transients have damped out (Kundur, 1994; Tanaka et al., 1994).

2.3 Model Structures and Parameters

The accuracy of stability analysis has significant impact on power system operational guidelines, operational planning, and design. Accurate stability analysis is necessary
to allow for more precise calculations of power transfer capability of transmission
grids. The accuracy of stability analysis, however, largely depends on the validity
of system models employed in describing power system dynamic behaviors
(here, system model refers to the model structure and its associated parameter
values). Accurate system models are essential for simulating complex power system
behaviors.

In the past, the issue of accurately modeling power system components such as
synchronous generators, excitation systems, and loads has received a great deal of
attention from the power industry. Standard generator and excitation model struc-
tures have been developed (IEEE Standard 421.1, 1986; IEEE Standard 421.5,
1992). The remaining issue is how to derive accurate parameter values for these
standard models. This issue is at the heart of parameter estimation in system
identification.

Manufacturers develop parameter values for the model structures of generators
and their control systems by using an “off-line” approach. In most cases, the param-
eter values provided by manufacturers are fixed and do not reflect the actual system
operating conditions. The effect of nonlinear interaction between the generator (or
control system) and the other parts of the system may alter parameter values. For
instance, when an excitation system is put into service, its model parameter values
may drift due to (1) changes in system operating conditions, (2) the nonlinear inter-
action between the excitation system and the rest of the power system, and (3) the
degree of saturation and equipment aging, and so on. Also, the parameter values of
excitation systems provided by manufacturers are typically derived from tests at the
plant, before the excitation system is actually put into service, and are often per-
formed by measuring the response of each individual component of the device sepa-
rately and then by combining those individual components to yield an overall system
model. Although adjustments can be made during commissioning, accurate param-
er values may not be generally available once the device is installed into the power
system. This prompts the use of an online measurement-based approach for develop-
ing accurate parameter values.

The measurement-based approach has the advantage of providing reliable data
for generators and their integrated control systems by directly measuring the system
dynamic behavior as a whole to yield accurate models. For instance, the task of
measurement-based generator parameter estimation is to accurately identify the
machine’s direct and quadrature resistances as well as reactances simultaneously
based on measurements without resorting to various off-line tests, such as the open
circuit test, in which the machine is isolated from the power system.

Compared to activities in the modeling of generators, loads, and excitation
systems, relatively little effort has been devoted to parameter estimation for gover-
nors and turbines, whose standard model structures have already been developed
(Hamnett et al., 1995). This may be explained by the fact that governors and turbines
play an important role in power system midterm or long-term stability, but not so
much in transient stability, which is much more widely scrutinized. The boiler
model, also more relevant in long-term stability studies, is not supported in most
current production-grade power system stability programs. In the case of HVDC and
some FACTS devices, no standard model is currently available. This is due to one or more of the following reasons: (1) the particular device is new, and standard controls are not well-defined; (2) the device occurs only rarely; and (3) each installation requires a different model.

2.4 MEASUREMENT-BASED MODELING

In the last 20 years, a significant amount of effort has been devoted to measurement-based parameter estimation for synchronous generators, excitation systems, and loads. These efforts are mostly based on the following two classes of methods for estimating these parameters:

- time-domain methods and
- frequency-domain methods.

Historically, frequency-domain methods have appeared to dominate the theory and practice of system identification in control engineering applications. Presently, the literature on system identification is very much dominated by time-domain methods. If the intended use of the model derived from the system identification procedure is to simulate the system or to predict the future outputs of the system, then time-domain methods are most appropriate. Similarly, if the derived model is to be used in conjunction with any state-space/time-domain control system design procedure, then again, time-domain methods are best. However, if the object of system identification is simply to gain general insight into the system, for instance, determining resonances in the response, then frequency-domain methods are probably most appropriate. Most IEEE standard model structures for power components are expressed in the time-domain.

The process of parameter estimation based on measurements can be summarized as follows:

1. Determine a set of input–output data derived from the physical system under study.
2. Estimate its parameters using a suitable method and an estimation criterion.
3. Validate the estimated model using the input–output data.
4. If unsatisfactory, try another model structure and repeat step 2, or try another identification method and another estimation criterion and repeat step 2 until a “satisfactory” model is obtained.

The data from the online measurement are obtained during the occurrence of power system disturbances such as line trippings and faults. The data so acquired reflect the intrinsic characteristics of the system components under normal operating conditions and can be utilized to obtain better parameter values. These improved parameters can in turn be used to improve the modeling and simulation of power system dynamics.

One challenging task in power system modeling is the load modeling. This manifests itself in the unavailability of standard load model structures even though
standard generator and excitation model structures have been developed and accepted in the power industry. It is well known that load behaviors have profound impacts on power system dynamic behaviors. Inaccurate load models, for instance, can lead to a power system being operated in modes that result in actual system collapse or separation (CIGRE Task Force 38.02.05, 1990). Simulation studies using simple load models were found to fail in explaining voltage collapse scenarios. Accurate load models are necessary to ensure simulation accuracy in grid operations and planning studies so that more precise stability limits can be determined.

Load models adequate for some types of power system dynamic analysis may be not adequate for others. Hence, representative load models should be developed for certain types, not all types of power system dynamic analysis. For example, voltage stability analysis is more concerned with dynamic behaviors of reactive loads, while transient stability analysis is more concerned with dynamic behaviors of real loads (Liang et al., 1998; Xu and Mansour, 1994). Load models for certain types of power system dynamic analysis were developed in Choi (2006), CIGRE Task Force 38.02.05 (1990), and Ju et al. (1996).

A load model is a mathematical representation of the relationship between a bus voltage (magnitude and frequency) and power (real and reactive) or current flowing into the bus load. At present, the so-called static load model structure (where the load is represented as constant impedance, constant current, constant MVA, or a combination of the three) or voltage frequency-dependent load structure is still commonly used in computer program power system analysis. These static load models are adequate for some types of power system dynamic analysis but not for others. There remains a necessity for the development of accurate dynamic load models. Because of its importance, the subject of load modeling has drawn significant research efforts, for example, those documented in Choi (2006), CIGRE Task Force 38.02.05 (1990), He et al. (2006), IEEE Task Force on Load Representation for Dynamic Performance (1993), and Ju et al. (1996). There are two main time-domain approaches available for developing accurate load models:

- component-based approach and
- measurement-based approach.

The component-based approach builds up the load model from information on dynamic behaviors of all the individual components (Price et al., 1988). Load composition data, load mixture data, and the dynamic behavior of each individual load component of a particular load bus are considered. For a large utility, such surveys of load components can be very difficult and cumbersome tasks.

The measurement-based approach involves placing measurement systems at load buses for which dynamic load models will be developed (CIGRE Task Force 38.02.05, 1990; Craven and Michael, 1983; Hiskens, 2001; Ju et al., 1996). This approach has the advantage of employing direct measurements of the actual load behaviors during system disturbances so that accurate load models can be obtained directly in the form needed for the inputs of existing power system analysis and control programs. These two approaches are complementary to each other. The component-based approach is useful for deriving a suitable model structure for a
load bus, whereas the measurement-based approach is appropriate for obtaining values for the associated model parameters.

Figure 2.3 shows a schematic description of the measurement-based load modeling approach. A procedure for identifying a load model using the measurement-based approach is described in the following:

**Step 1.** Obtain a set of input–output data derived from a set of measurements.

**Step 2.** Select a load model structure.

**Step 3.** Estimate its parameters using a suitable method and estimation criterion.

**Step 4.** Validate the derived model with the parameters obtained in Step 3.

**Step 5.** If the validation criterion is not met, take remedy actions; for example, try another estimation method or try another model structure and go to Step 3.

### 2.5 POWER SYSTEM STABILITY PROBLEMS

It is fair to state that any system may always present instabilities when sufficiently large disturbances are introduced. The key point is to find the “proper” disturbance and the appropriate stability condition when a given system or phenomenon is investigated (Hahn, 1967; IEEE TF Report, 1982; Kundur, 1994). A proper disturbance should be relevant to the system and physically meaningful. The “appropriate” stability condition is concerned with the range of deviation in the state space.

There are two types of disturbances in power systems: event disturbances and load variations (Balu et al., 1992). The fundamental problem of power system stability analysis relative to a disturbance (i.e., fault) can be broadly stated as follows:
given a prefault SEP and a fault-on system, will the postfault trajectory settle down to an acceptable SEP (Varaiya et al., 1985)? If the postfault trajectory converges to an equilibrium point, then the system is said to be (rotor) angle stable relative to the fault. Physically, the angle stability represents the ability of synchronous generators of an interconnected power system to remain synchronized after a fault, which initially creates electromechanical oscillations between generators in the system. Furthermore, if the voltage magnitudes both at the equilibrium point and along the postfault trajectory are acceptable, then the system is said to be voltage stable relative to the fault.

Power system instability occurs when the postfault trajectory of a power system subject to a fault does not approach an acceptable steady state (or equilibrium point). Depending on the behaviors of the postfault trajectory, power system instability may be manifested in several different ways. The type of instability may be angle related (an angle instability) or voltage related (a voltage instability), or both. If the voltage magnitudes along the postfault trajectory are unacceptable or the postfault trajectory approaches a steady state with unacceptable voltage magnitudes, then the system is said to suffer from voltage instability relative to the fault. Another type of power system instability is the emergence of oscillatory behaviors such as SSR and low-frequency oscillations. Oscillatory behaviors occur when the postfault trajectory either experiences a prolonged oscillatory transient or approaches a stable periodic solution.

Depending on the size of a disturbance, power system stability can be classified into small-signal (small-disturbance) stability and general (large-disturbance) stability. Small-signal stability refers to the ability of the power system (at an equilibrium point) to maintain synchronism under small disturbances. Given a power system model, a disturbance is called small if the stability of the system can be analyzed based on a linear system model obtained from linearizing the original nonlinear model around the equilibrium point; otherwise, it is called large. Small disturbances, such as small variations in loads and real power generations, occur continually on power systems.

Small-signal stability can be analyzed based on the eigenvalues of the Jacobian matrix at the equilibrium point. Most event disturbances and events involving large variations in loads and real power generations are considered large disturbances. Large-disturbance stability needs to be analyzed based on the postfault trajectory of the original system model. An equilibrium point (operating point) of a power system is small-signal stable if it is an asymptotically stable equilibrium point. From the viewpoint of nonlinear stability analysis, large-disturbance stability, such as angle/voltage stability and transient angle/voltage stability, belongs to the class of asymptotic stability problems.

Two types of basic information are needed in order to perform a power system stability analysis: static data (which is the power flow data) and dynamic data. The power flow data describe the network and its steady-state operating conditions. The dynamic data supply the information needed to compute the response of the modeled devices to a given disturbance. The dynamic data include models of detailed synchronous machines, dynamic load models, induction motor, static VAR compensa-
2.6 Approaches for Stability Analysis

The complete model for the stability analysis of a typical power system is a set of DAEs numbering on the order of thousands or tens of thousands. At present, stability analysis programs routinely used in utilities around the world are mostly based on step-by-step numerical integrations of the set of equations to simulate system behaviors relative to a given disturbance. A numerical solution of a set of DAEs is a sequence of approximate values of the solution function (curve) at a discrete set of points. A general overview of step-by-step numerical integration methods for power system stability analysis appears in Stott (1979). Several production-grade time-domain simulation packages described in the literature can be found, for example, in Electric Power Research Institute (1992), Kurita et al. (1993), de Mello et al. (1992), Stubbe et al. (1989), and Tanaka et al. (1994).

An alternative approach to transient stability analysis employing energy functions, called direct methods, was originally proposed by Magnusson in 1947 and was pursued in the 1950s and 1960s by several researchers, for example, Aylett (1958), Gless (1966), and El-Abiad and Nagappan (1966). Direct methods received significant research efforts in the 1970s. Direct methods have had a long developmental history spanning six decades, but until recently, many were thought to be impractical for large-scale power systems. Among the direct methods, the classical method of using the concept of closest unstable equilibrium point (UEP) gives a very conservative assessment of stability. The potential energy boundary surface (PEBS) method is fast but may give inaccurate stability assessments. The controlling UEP method is the most viable direct method in terms of its accuracy and slightly conservative nature. The success of the controlling UEP method, however, hinges on...
the ability to compute the controlling UEP. A great majority of work on the controlling UEP method is based on physical reasonings, heuristics, and simulations without theoretical support. This work has achieved limited success in computing the controlling UEP.

Recent developments of the family of boundary of stability region controlling unstable equilibrium point (BCU) methods have revived the controlling UEP method. The BCU method can reliably compute the controlling UEP. The combination of the controlling UEP method and the BCU method now emerges as a practical means for solving large-scale transient stability analysis problems. Extensive evaluations of the BCU-based controlling UEP method on large-scale power systems such as a 12,000-bus power system were conducted and promising results were reported (Chiang et al., 2006; Tada et al., 2005). Several major installations of BCU methods in modern energy management systems at large power companies have taken place.

The controlling UEP method provides several advantages in stability assessment and control. It can determine transient stability without the time-consuming numerical integration of the (postfault) power system. In addition to its speed, the controlling UEP method also provides an accurate quantitative measure of the degree of system stability. This additional information makes the controlling UEP method very attractive when the relative stability of different network configuration plans must be compared or when system operating limits constrained by transient stability must be calculated quickly. Another advantage of the controlling UEP method is the ability to provide useful information regarding how to derive preventive control actions when the underlying power system is deemed unstable and how to derive enhancement control actions when the underlying power system is deemed critically stable.

We next describe these two different approaches from the state-space viewpoint. The time–domain approach computes the relationship between the prefault stable equilibrium point and the ultimate postfault state explicitly, using step-by-step numerical integrations to simulate the entire system trajectory. On the other hand, direct methods use two steps to solve the stability problem. First, they compute only the relationship between the prefault SEP and the system state at the time of fault clearing using the step-by-step numerical integration of the fault-on system. In the second step, the direct methods directly determine, without numerical integration of the postfault system, whether the initial state of the postfault system lies inside the stability region of a desired SEP. This direct determination of the stability property is based on an energy function (defined for the postfault system) and on a critical energy (relative to the fault-on trajectory). It will be shown that if the energy function value of the initial state of the postfault system is less than the critical energy, then the postfault trajectory will settle down to the desired postfault SEP. This is the analytical basis of direct methods. The great challenge in direct methods is determining the critical energy relative to a fault-on trajectory and deriving energy functions for power system stability models. A comparison between the time–domain simulation approach and the direct methods is summarized in Table 2.2.
Table 2.2 A Comparison of the Time–Domain Approach and the Direct Methods

<table>
<thead>
<tr>
<th></th>
<th>Time–domain approach</th>
<th>Direct methods (using energy function)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prefault system</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fault-on system</td>
<td>$x = f_s(x, y)$</td>
<td>$0 = g_p(x, y)$</td>
</tr>
<tr>
<td></td>
<td>$t_0 \leq t \leq t_c$</td>
<td></td>
</tr>
<tr>
<td><strong>Prefault SEP</strong></td>
<td>$x(t)$</td>
<td>$x(t_0)$</td>
</tr>
<tr>
<td></td>
<td>End point of fault-on trajectory</td>
<td>Fault-on trajectory</td>
</tr>
<tr>
<td><strong>Fault-on system</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x = f(x)$</td>
<td>$0 = g(x, y)$</td>
</tr>
<tr>
<td></td>
<td>$t_c \leq t \leq t_\infty$</td>
<td></td>
</tr>
<tr>
<td><strong>Postfault system</strong></td>
<td>$x(t)$</td>
<td>$x(t)$</td>
</tr>
<tr>
<td></td>
<td>Initial point of postfault trajectory</td>
<td>Postfault trajectory</td>
</tr>
<tr>
<td><strong>Computation</strong></td>
<td>Numerical integration methods to derive the fault-on trajectory</td>
<td>Numerical integration methods to derive the fault-on trajectory</td>
</tr>
<tr>
<td><strong>Postfault trajectory</strong></td>
<td>$t = t_0$</td>
<td>$t = t_0$</td>
</tr>
<tr>
<td></td>
<td>$t = t_f$</td>
<td>$t = t_f$</td>
</tr>
<tr>
<td></td>
<td>Postfault trajectory</td>
<td>Postfault trajectory</td>
</tr>
</tbody>
</table>

2.7 CONCLUDING REMARKS

After decades of research and developments in the energy function-based direct methods and the time–domain simulation approach, it has become clear that the capabilities of direct methods and that of the time–domain approach complement each other. The current direction of development is to include appropriate direct methods, such as the BCU-based controlling UEP method and a fast time–domain simulation program within the body of overall power system stability simulation programs (Chiang et al., 2007; Jardim et al., 2004; Kim, 1994; Tada and Chiang, 2008). In this development, the BCU-based controlling UEP method provides the advantages of fast computational speed and energy margins, which make it a good complement to the traditional time–domain simulation approach.

The controlling UEP and its functional relations to certain power system parameters, such as power injections, are an effective complement to develop fast
calculators for available transfer capability limited by transient stability. In addition, they provide effective information to develop the following control actions:

1. preventive control schemes for credible contingencies that are unstable,
2. enhancement control schemes for credible contingencies that are critically stable, and
3. enhancement control for increasing available transfer capability limited by transient stability.

The analytical basis for direct methods and, in particular, the controlling UEP method, is an understanding of stability regions of nonlinear dynamical systems. We will provide a rigorous theoretical foundation for the direct methods, including the closest UEP method, the PEBS method, and the controlling UEP method in Chapters 8 through 13. The central theme of the theoretical foundation is knowledge of the stability region, which will be developed in the next three chapters.
Chapter 3

Lyapunov Stability and Stability Regions of Nonlinear Dynamical Systems

3.1 INTRODUCTION

Stability is a fundamental subject that unifies engineering and the sciences. This subject has been regarded by many as a fascinating and difficult problem of human culture. As such, there are at least 50 different terms for stability concepts used in the literature. Stability is a very broad subject, and the concept of stability can be formulated in a variety of ways depending on the intended use of stability analysis and design (Alberto and Chiang, 2007; DeCarlo et al., 2000; Hahn, 1967; La Salle and Lefschetz, 1961; May, 1973; Michel et al., 1982). In this chapter, we review some relevant stability concepts from the nonlinear dynamical systems theory. Some of these concepts and their implications will be discussed in detail.

Another important subject related to stability is the stability region of nonlinear dynamical systems. Indeed, the problem of determining stability regions (regions of attraction) of nonlinear dynamical systems is of fundamental importance for many disciplines in engineering and in the sciences (Athay et al., 1984; Chiang et al., 1988; Genesio et al., 1985; Loparo and Blankenship, 1978; Saha et al., 1997; Sastry, 1999; Vu and Liu, 1992). For instance, knowledge of the stability region is essential in the development of direct methods for power system transient stability analysis. Indeed, this problem is at the heart of direct methods. A comprehensive theory of stability regions for the (autonomous) dynamical system will be derived in this chapter. Most of the proofs presented in this chapter are taken from Chiang et al. (1987, 1988) and from Chiang and Thorp (1989b). The topic of estimating stability regions for higher-dimension nonlinear dynamical systems will be discussed in Chapter 5.
3.2 EQUILIBRIUM POINTS AND LYAPUNOV STABILITY

We consider the following (autonomous) nonlinear dynamical system:

$$\dot{x} = f(x), \ x \in \mathbb{R}^n.$$  \hspace{1cm} (3.1)

It is natural to assume the function (i.e., the vector field) $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies a sufficient condition for the existence and uniqueness of a solution. The solution curve of Equation 3.1 starting from $x$ at $t = 0$ is called the system trajectory starting from $x$ and is denoted by $\phi(x, \cdot)$. The system trajectory starting from $x$ is a function of time; given a specified time, the system trajectory function maps the specified time into a point in the state space. $\bar{x} \in \mathbb{R}^n$ is said to be an equilibrium solution of Equation 3.1 if $f(\bar{x}) = 0$; that is, the equilibrium point is a solution that does not change in time. Hence, equilibrium points are degenerated solution curves that do not move.

The stability property of an equilibrium point will be discussed next. We first review the concepts of (Lyapunov) stability and asymptotic stability.

Definition: Lyapunov Stability
An equilibrium point $\bar{x} \in \mathbb{R}^n$ of Equation 3.1 is said to be (Lyapunov) stable if, for each open neighborhood $U$ of $\bar{x} \in \mathbb{R}^n$, there exists an open neighborhood $V$ of $\bar{x} \in \mathbb{R}^n$ such that, for all $x \in V$ and for all $t > 0$, $\phi_t(x) \in U$. Equivalently, $\phi_t(x) \in U$ for all $t > 0$. Otherwise, $\bar{x}$ is unstable.

The concept of (Lyapunov) stability is illustrated in Figure 3.1. Intuitively speaking, an equilibrium point is stable if nearby trajectories stay nearby. In many applications, the requirement of “nearby trajectories stay nearby” is not sufficient; instead, the requirement becomes “nearby trajectories stay nearby and all converge to the equilibrium point.” Under this situation, the concept of (Lyapunov) stability can be sharpened into the concept of asymptotic stability as defined in the following.

Definition: Asymptotic Stability
An equilibrium point $\bar{x} \in \mathbb{R}^n$ of Equation 3.1 is said to be asymptotically stable if there exists an open neighborhood $U$ of $\bar{x} \in \mathbb{R}^n$, such that, for all $x \in U$ and for all
3.2 Equilibrium Points and Lyapunov Stability

$t > 0, \phi_t(x) \in U$. In addition, every trajectory $\phi_t(x)$ starting from this neighborhood converges to the equilibrium point $\bar{x}$.

Equivalently,

(i) $\phi_t(x) \in U, t > 0$ and

(ii) $\lim_{t \to \infty} \|\phi_t(x) - \bar{x}\| = 0$.

The concept of asymptotic stability is illustrated in Figure 3.2. Intuitively speaking, an equilibrium point is asymptotically stable if it is the sink of nearby trajectories. An asymptotically stable equilibrium point is also termed a sink in the classical literature of nonlinear dynamical systems (Athay et al., 1984; Guckenheimer and Holmes, 1983; Hirsch and Smale, 1974).

In order to determine the stability of $\bar{x}(t)$, we must understand the nature of solutions near $\bar{x}(t)$. Let

$$x(t) = \bar{x}(t) + y(t).$$  \hspace{1cm} (3.2)

Substituting Equation 3.2 into Equation 3.1 and Taylor expanding about $\bar{x}(t)$ gives

$$\dot{x}(t) = \dot{\bar{x}}(t) + \dot{y} = f(\bar{x}(t) + Df(\bar{x}(t))y + O(\|y\|^2))$$  \hspace{1cm} (3.3)

where $Df$ is the derivative of $f$ and $\|\|$ denotes a norm on $\mathbb{R}^n$. Using the fact that $\bar{x}(t) = f(\bar{x}(t))$, Equation 3.3 becomes

$$\dot{y} = Df(\bar{x}(t))y + O(\|y\|^2).$$  \hspace{1cm} (3.4)

Equation 3.4 describes the evolution of orbits near $\bar{x}(t)$. For stability questions, we are concerned with the behavior of solutions arbitrarily close to $\bar{x}(t)$, so it seems reasonable that this question could be answered by studying the associated linear system:

$$\dot{y}(t) = Df(\bar{x}(t))y.$$  \hspace{1cm} (3.5)

Therefore, the question of stability of $\bar{x}(t)$ involves the following two steps:

1. Determine if the $y = 0$ solution of Equation 3.5 is stable.

2. Show that stability (or instability) of the $y = 0$ solution of Equation 3.5 implies stability (or instability) of $\bar{x}(t)$ of the system (Eq. 3.1).
It can be shown that if the eigenvalues of the associated linear vector field have nonzero real parts, then the orbit (i.e., trajectory) structure near an equilibrium solution of the nonlinear vector field (Eq. 3.1) is essentially the same as that of the linear vector field (Eq. 3.5). This is addressed by the following theorem.

**Theorem 3.1: Hartman–Grobman (Hirsch and S. Smale, 1974)**

Consider the system (Eq. 3.1) with an equilibrium point $\bar{x}$. If $Df(\bar{x})$ has no zero of purely imaginary eigenvalues, there is a homeomorphism $h$ defined on a neighborhood $U$ of $\bar{x}$ taking orbits of the flow $\phi_t$ to those of the linear flow $e^{tDf(\bar{x})}$ of Equation 3.5. The homeomorphism preserves the sense of the orbits and is chosen to preserve parameterization by time.

A corollary of Theorem 3.1 and a fundamental linear system theory lead to the following sufficient condition for an equilibrium point to be asymptotically stable.

**Theorem 3.2: Asymptotic Stability (Hirsch and Smale, 1974; Sastry, 1999)**

Suppose all of the eigenvalues of $Df(\bar{x})$ of the linear system (Eq. 3.5) have negative real parts. Then, the equilibrium solution $x = (\bar{x})$ of the nonlinear system (Eq. 3.1) is asymptotically stable.

This is a well-known stability evaluation method that checks the eigenvalues of the Jacobian matrix at an equilibrium point. The required computational efforts can be immense. An alternative approach is based on the Lyapunov function theory, which is discussed in the next section.

### 3.3 Lyapunov Function Theory

One of the most important developments in the stability theory is the Lyapunov function theory. Lyapunov, a Russian mathematician and engineer, laid down the foundation for the Lyapunov theory. Lyapunov stability theorems give sufficient conditions for Lyapunov stability and asymptotic stability. The appealing feature of the Lyapunov function theory is that it derives the stability properties of the equilibrium point without numerically solving the underlying ordinary differential (difference) equations. This is the “spirit of Lyapunov.” There are several versions of proof of the Lyapunov function theorem.

During the early period of developing direct methods, the Lyapunov function theorem was applied by many researchers to ensure power system transient stability without time-domain simulation. In this section, we present an overview of a fundamental Lyapunov function theorem.

We denote the following notation as the time derivative of a function $V(x)$, taken along the system trajectory:
3.3 Lyapunov Function Theory

\[ \dot{V}(x(t)) = \frac{\partial V(x(t))^T}{\partial x} \cdot \dot{x}(t) = \frac{\partial V(x)^T}{\partial x} \cdot f(x). \] (3.6)

Since the vector field \( f(x) \) and the gradient of the function \( V(x) \) are available without the explicit knowledge of the system trajectory, the time derivative of \( V(x(t)) \) can be performed without knowledge of the system trajectory.

**Theorem 3.3: Lyapunov’s Stability** (Guckenheimer and Holmes, 1983; Hirsch and Smale, 1974)

Let \( \hat{x} \) be an equilibrium point of \( \dot{x} = f(x) \), where \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \). Let \( V: U \rightarrow \mathbb{R} \) be a continuous function defined on a neighborhood of \( \hat{x} \), differentiable on \( U \rightarrow \hat{x} \), such that

(a) \( V(\hat{x}) = 0 \) and \( V(x) > 0 \) if \( x \neq \hat{x} \), and \( x \in U \).

(b) \( \dot{V}(x) \leq 0 \) in \( U - \hat{x} \).

Then, \( \hat{x} \) is stable. Furthermore, if also

(c) \( \dot{V}(x) < 0 \) in \( U - \hat{x} \), then \( \hat{x} \) is asymptotically stable.

(d) \( \dot{V}(x) < 0 \) in \( U - \hat{x} \), \( U = \mathbb{R}^n \), then \( \hat{x} \) is asymptotically stable in the large.

**Remarks:**

1. The Lyapunov function theory asserts not only the stability property of the equilibrium point (a local result) but also that there does not exist any limit cycle (oscillation behavior) or bounded complicated behavior such as an almost periodic trajectory, chaotic motion, and so on, in the subset of the state space where there exists a Lyapunov function.

2. It should be pointed out that the Lyapunov function theory only furnishes sufficient conditions. If for a particular Lyapunov function candidate, \( V \), the required conditions on the derivative of \( V \), that is, \( \dot{V} \), are not met, then conclusions regarding the stability or instability of the equilibrium point still cannot be drawn.

There is no systematic way of constructing Lyapunov functions for general nonlinear systems. This is a fundamental drawback of the Lyapunov direct method. Therefore, faced with specific nonlinear dynamical systems, one often has to use experience, intuition, trial and error, and physical insights (e.g., the energy function for electrical and mechanical systems) to search for an appropriate Lyapunov function. In this literature, a number of methods and techniques facilitating the search of Lyapunov functions have been proposed (Khalil, 2002; Michel et al., 1984; Vaahedi et al., 1998; Vidyasagar, 2002).
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3.4 STABLE AND UNSTABLE MANIFOLDS

The concepts of an invariant set, a limit set including an \( \alpha \)-limit set and an \( \omega \)-limit set, and stable and unstable manifolds are important to the dynamical system theory. Each of these concepts is defined next. A detailed discussion of these concepts and implications may be found in Guckenheimer and Holmes (1983), in Paganini and Lesieutre (1999), and in Palis and de Melo (1981).

A set \( M \in \mathbb{R}^n \) is called an invariant set of Equation 3.1 if every trajectory of Equation 3.1 starting in \( M \) remains in \( M \) for all \( t \). A point \( p \) is said to be in the \( \omega \)-limit set of \( x \) if, corresponding to each \( \varepsilon > 0 \) and \( T > 0 \), there is a \( t > T \) with the property that \( |\phi(x, t) - p| < \varepsilon \). This is equivalent to saying that there is a sequence \( \{t_i\} \) in \( \mathbb{R} \), \( t_i \to \infty \), with the property that \( p = \lim_{i \to \infty} \phi(x, t_i) \). A point \( p \) is said to be in the \( \alpha \)-limit set of \( x \) if, corresponding to each \( \varepsilon > 0 \) and \( T < 0 \), there is a \( t < T \) with the property that \( |\phi(x, t) - p| < \varepsilon \). This is equivalent to saying that there is a sequence \( \{t_i\} \) in \( \mathbb{R} \), \( t_i \to -\infty \), with the property that \( p = \lim_{i \to -\infty} \phi(x, t_i) \). Hence, the \( \omega \)-limit set captures the asymptotic behaviors of a bounded trajectory in positive time, while the \( \alpha \)-limit set captures the asymptotic behaviors of a bounded trajectory in negative time.

One of the fundamental properties of the limit set is as follows.

Theorem 3.4: Properties of Limit Sets

If a trajectory \( \phi(x, t) \) of a system (Eq. 3.1) is bounded for \( t \geq 0 \) (or \( t \leq 0 \)), then its \( \omega \)-limit set (or \( \alpha \)-limit set) exists; moreover, its limit set is compact, connected, and invariant.

Generally speaking, limit sets can be very complex; they can be equilibrium points, limit cycles (closed orbits), quasi-periodic solutions, and chaos. Stable limit sets are of supreme importance in experimental and numerical settings because they are the only kind of limit sets that can be observed naturally! The concept of stable limit sets is similar to that of equilibrium points.

Definition: Stable Limit Set

A limit set \( L \) is said to be Lyapunov stable if, for each open neighborhood \( U \) of \( L \), there exists an open neighborhood \( V \) of \( L \), such that for all \( x \in V \) and for all \( t > 0 \), \( \phi(x) \in U \). Equivalently, \( \phi(x) \in U \) for all \( t > 0 \). Otherwise, \( L \) is unstable.

Definition: Asymptotically Stable Limit Set

A limit set \( L \) is asymptotically stable if there exists an open neighborhood \( V \) of \( L \) such that the \( \omega \)-limit set of every point in \( V \) is \( L \). Equivalently,

\[
\begin{align*}
(\text{i}) & \quad \phi(x) \in V \text{, } t > 0 \text{ and } \\
(\text{ii}) & \quad \lim_{t \to \infty} \inf_{y \in L} \|\phi_t(x) - y\| = L.
\end{align*}
\]

We next review the concept of stable and unstable manifolds of limit sets. We start from the simplest limit set—the equilibrium point. Let \( \hat{x} \) be an equilibrium point...
and $U \subset \mathbb{R}^n$ be a neighborhood of $\hat{x}$. We define the local stable manifold of $\hat{x}$ as follows:

$$W_{loc}^s(\hat{x}) := \{ x \in U : \phi_t(x) \to \hat{x} \text{ as } t \to \infty \}.$$ 

The local unstable manifold of $\hat{x}$ is defined as

$$W_{loc}^u(\hat{x}) := \{ x \in U : \phi_t(x) \to \hat{x} \text{ as } t \to -\infty \}.$$ 

Note that $W_{loc}^s(\hat{x})$ is a positive invariant set, while $W_{loc}^u(\hat{x})$ is a negative invariant set. They may not be manifolds when $\hat{x}$ is nonhyperbolic. We next present a fundamental theorem of stable and unstable manifolds for a hyperbolic equilibrium point.

Recall that an equilibrium point is hyperbolic if the corresponding Jacobian matrix has no eigenvalues with zero real parts; otherwise, it is a nonhyperbolic equilibrium point.

**Theorem 3.5: Unstable–Stable Manifold Theorem**

(Guckenheimer and Holmes, 1983; Sastry, 1999)

Suppose that the nonlinear system (Eq. 3.1) has a hyperbolic equilibrium point $\bar{x}$. Then the sets $W_{loc}^s(\bar{x})$ and $W_{loc}^u(\bar{x})$, often referred to as the local stable and unstable manifolds, are manifolds of the same dimensions $n_s, n_u$ as those of the stable and unstable eigenspaces $E^s, E^u$ of the linearized system (Eq. 3.5). These manifolds are also tangent to $E^s, E^u$ at $\bar{x}$. $W_{loc}^s(\bar{x})$ and $W_{loc}^u(\bar{x})$ are as smooth as the vector field $f(x)$ of Equation 3.1 (see Figure 3.3).

The stable manifold $W^s(\bar{x})$ and the unstable manifold $W^u(\bar{x})$ are obtained by letting points in $W_{loc}^s(\bar{x})$ flow backward in time and points in $W_{loc}^u(\bar{x})$ flow forward in time:

$$W^s(\bar{x}) := \bigcup_{t \leq 0} \phi_t(W_{loc}^s(\bar{x})) \quad \text{and} \quad \tag{3.7}$$

$$W^u(\bar{x}) := \bigcup_{t \geq 0} \phi_t(W_{loc}^u(\bar{x})). \quad \tag{3.8}$$

![Figure 3.3](image_url) The local stable and unstable manifolds of an equilibrium point.